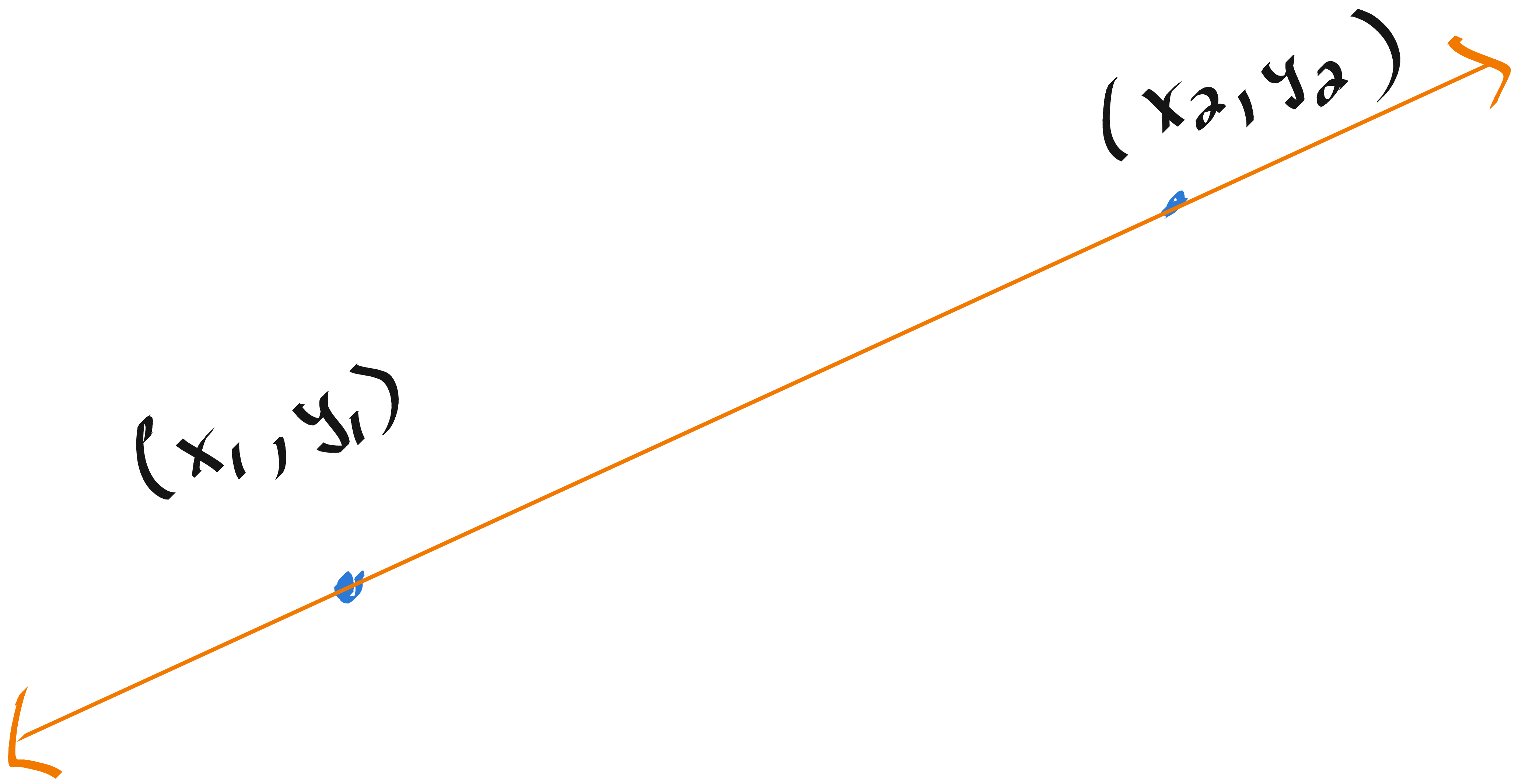
Interpolating Polynomials

" Through any two points, there is a <u>line</u>. "

First point: (x_1, y_1) pair of numbers

Second point: (x_2, y_2)



If $x_1 \neq x_2$ the line you get out is of the form $y = mx + b$, i.e., a function of x.

Q: What about 3 points?

What I want: Some polynomial $p(x)$ that passes through 3 given points. We need the x-values of the points to all be different.

Example 1: Given the points $(-2,3)$, $(1,4)$, and $(3,8)$, find a polynomial that passes through all 3 points.

Solution: Can we solve this easily, i.e., is the polynomial a line?

Slope between $(1,4)$ and $(3,8)$:

$$\frac{8-4}{3-1} = 2$$

Slope between $(-2,3)$ and $(1,4)$:

$$\frac{4-3}{1-(-2)} = \frac{1}{3} \neq 2$$

There's no way to make a line through these three points.

What about a quadratic

$$y = p(x) = ax^2 + bx + c$$

We're finding the coefficients a, b, and c.

Plug in our points.

$$\boxed{\begin{aligned} 3 &= p(-2) = 4a - 2b + c \\ 4 &= p(1) = a + b + c \\ 8 &= p(3) = 9a + 3b + c \end{aligned}}$$

system of linear equations

$$3 = 4a - 2b + c$$

$$4 = a + b + c$$

$$8 = 9a + 3b + c$$

Can add/subtract multiples of one equation from another without changing the solution.

Subtract 1^{st} equation from the 2^{nd}:

$$1 = -3a + 3b$$

Subtract 2^{nd} equation from the 3^{rd}:

$$4 = 8a + 2b \text{, reduce to}$$

$$2 = 4a + b$$

We now have 2 equations in just "a" and "b":

$$1 = -3a + 3b$$

$$2 = 4a + b$$

solve for b, plug into 1st equation

$$b = -4a + 2$$

$$1 = -3a + 3(-4a + 2)$$

$$-5 = -15a$$

$$a = 1/3$$

plug in

$$b = -4a + 2 = -4(1/3) + 2$$

$$= 2/3$$

Finally, use any of our original 3 equations to solve for c.

$$4 = a + b + c$$

$$4 = 1/3 + 2/3 + c$$

$$c = 3$$

$$a = 1/3, \quad b = 2/3, \quad c = 3$$

$$\boxed{y = \frac{1}{3}x^2 + \frac{2}{3}x + 3}$$ works!

Q: Suppose you have two points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$.

How many quadratics pass through these points?

Infinitely many!

Example 2: Given the points $(5,7)$ and $(-8,2)$, how many quadratics pass through these two points?

Solution: as before, write the quadratic as $p(x) = ax^2 + bx + c$ and plug in the points.

$$7 = p(5) = 25a + 5b + c$$

$$2 = p(-8) = 64a - 8b + c$$

$$7 = 25a + 5b + c$$

$$2 = 64a - 8b + c$$

Subtract 2^{nd} equation from the first:

$$5 = -39a + 13b$$

$$13b = 5 + 39a$$

$$b = \frac{5 + 39a}{13}$$

a can be any nonzero number!

Q: Given four points (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4). Can we always find a quadratic that passes through these four points if none of the x-values equal eachother?

No.

Example 3: Given the points $(-2,3)$, $(1,4)$, $(3,8)$, and $(5,-1)$, Show there is no quadratic through all four points.

Solution: We already know from Example 1 that

$$p(x) = \frac{x^2}{3} + \frac{2x}{3} + 3$$

passes through the first three pairs of points. Just plug in $x=5$ into this quadratic and check to see if you get -1 as the output.

$$p(5) = \frac{5^2}{3} + \frac{2}{3} \cdot 5 + 3$$

$$= \frac{25}{3} + \frac{10}{3} + 3 \neq -1$$

So there is no quadratic through these four points!

Note: if $p(5)$ had equalled -1, then the answer would have been yes.

Summary:

1) For 3 points, not all on a line, we can find a <u>unique</u> quadratic that passes through all 3 points if all the x-values of the points are different.

2) For 2 points, we can find <u>infinitely many</u> quadratics that pass through these points if the x-values are different.

3) For 4 points, either there is a unique quadratic or no quadratic provided the x-values are all different.

Moral: We either get 0, 1, or infinitely many solutions.

This is because <u>all</u> systems of linear equations have either 0, 1, or infinitely many solutions.

Example 4: Given the points $(-2,3)$, $(1,4)$, $(3,8)$, and $(5,-1)$, show there is a cubic through all four points.

Solution: Let $p(x) = ax^3 + bx^2 + cx + d$.

Plug in the points.

$$\underbrace{\begin{aligned} 3 &= p(-2) = -8a + 4b - 2c + d \\ 4 &= p(1) = a + b + c + d \\ 8 &= p(3) = 27a + 9b + 3c + d \\ -1 &= p(5) = 125a + 25b + 5c + d \end{aligned}}_{\text{System of linear equations}}$$

Solve the system like we did before. Or, ask Wolfram Alpha

Is there a better way to do this problem by hand? What is Wolfram Alpha doing?